Induction Puzzles and Perfect Information

LAMC Junior Circle

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1. **Taj and his Aunt:** Taj is on his way to visit his aunt, who lives across the state. It's her birthday, and he wants to bring her two cakes, one for her and one for his uncle. But, there are toll bridges between Taj and his aunt's house. Taj has no money, but he knows the hungry workers can be bribed. At each tollbooth, Taj can bribe the operator by giving him half of the total cakes Taj is carrying. Since the workers are kind, each operator will give Taj back one of his cakes after taking half of his supply.

   (a) How many cakes does Taj need to bring originally if there is only one tollbooth between him and his aunt? Remember, Taj must arrive at his aunt's house with two cakes.

   (b) How many cakes does Taj need to bring if there are two tollbooths? Three?

   (c) Extend this pattern. How many cakes would Taj need to bring if he needed to pass through 100 tollbooths?

   (d) Suppose Taj moved to a new state, where the tollbooth operators are much greedier. Here, they will only accept two-thirds of Taj's cakes as a bribe, but they will still return one cake afterward. Now how many cakes must Taj bring if there are five tollbooths between him and his aunt?

You may have gotten an answer you didn't expect in parts (a)-(c) of Problem 1. At first, part (c) would seem very difficult. But by starting at the end of the problem, from what a solution must look like and working backwards, the answer makes itself obvious! Let's try to continue this type of thinking with a different kind of problem:

2. **The King's Wise Men:** The King called the three wisest men in the country to his court to decide who would become his new advisor. He placed a hat on each of their heads, such that each wise man could see all of the other hats, but none of them could see their own. Each hat was either white or blue. The king gave his word to the wise men that at least one of them was wearing a blue hat - in other words, there could be one, two, or three blue hats, but not zero. The king also announced that the contest would be fair to all three men. The wise men were also forbidden to speak to each other. The king declared that whichever man stood up first and announced the color of his own hat would become his new advisor. The wise men sat for a very long time before one stood up and correctly announced the answer.

   (a) Imagine you are one of the wise men. Is it possible that the other two wise men are wearing white hats? Why or why not? Keep in mind these men are very intelligent, and it took one of them a long time to reason it out.

   (b) Can there be exactly one blue hat among the three? Why or why not?

   (c) Can there be exactly two blue hats among the three? Why or why not?

   (d) What was the correct answer, and how did he figure it out?

Just like in Problem 1, using induction and working backwards helps us find our solution. When we're dealing with people who act perfectly logically, like wise men, we must be able to follow their logical reasoning from beginning to end. But, as we've found, it's a bit easier to follow it from end to beginning. In fact, the most important piece of information in the problem was that no wise man immediately knew the answer. Each wise man knows what obvious situations would give them enough information to answer the question. Once they realize nobody is answering the question, that tells them no obvious situation has occurred. The key to solving these types of problems is realizing exactly that: if the characters don't have enough information to make a conclusion, that gives **you** more information about the problem.

3. **Josephine's Pet Kingdom:** In Josephine's Kingdom every citizen has to pass a logic exam before adopting a pet. A special strain of dog flu has entered the kingdom that also invisible to only the dog's owner. Every dog-owner knows the health-status of every other dog in the kingdom, except their own. But, they're each so scared of getting their own dog sick, that they will not tell the owner of a sick dog of the dog's sickness, so each owner must figure out themselves if their dog is healthy or ill. Each house has been given a special emergency light which can be seen by every other house in the kingdom. Queen Josephine announced that sick dogs had been discovered in the Kingdom, and if any dog-owner discovers their dog is sick, they must turn on their emergency light at midnight the day immediately following the discovery. For example, if I find out my dog is sick at 12:01AM, I will turn on my light at 12:00AM the next day, 23 hours and 59 minutes later.

   (a) What would happen if there were only one sick dog in the kingdom? Would the owner be able to figure it out? Why or why not?

   (b) How would the situation change if there were two sick dogs in the kingdom? How many days would it take for both of the sick dogs to be discovered?

   (c) Suppose there are n sick dogs. At what point (after how many days) will their owners realize they are sick?

These past two problems are examples of situations with **imperfect** information, as none of the wise men know the colors of the hats and none of the owners know if their dog is sick. Now, we will solve a problem in which everyone has **perfect** information. When everyone knows everything about the situation, and everyone acts in their own best interest, some interesting results can surface.

4. **Pirates and Gold:** Five pirates discover 100 pieces of gold. The Pirates have a ranking: Ahab is the captain, Blackbeard is second in command, Charles is third, David is fourth, and Edmond is fifth. To split the coins, they use the following process from the Pirate's Constitution. Of the men on the ship, the man of highest rank will propose a plan to split the coins among the pirates currently on board. Then, all pirates on board will vote on the plan. In the case of a tied vote, the plan succeeds, and the coins are split. If the vote fails, the pirate of highest rank who suggested the plan is thrown overboard, and the process repeats with the remaining pirates. All pirates are assumed to act perfectly logically with the goal of maximizing their personal profit. That means a pirate would vote for a plan where he gets just one coin rather than vote no and end up in a situation where he predicts he will get zero coins.

   (a) What would happen if there were only two pirates on board? What plan would the higher-rank pirate propose, and why?

   (b) Use your induction skills to determine Captain Ahab's optimal strategy. What plan should he propose? Remember, he wants to get as many coins as possible, but he also wants to be sure that he will not be thrown overboard.

BONUS: Intro to Mathematical Induction Great job finishing the packet! All the rest of this is extra material for you if you've finished. Now that we have a new vocabulary word, induction, we're going to learn a little bit more about this principle.

Induction means to extend a property, or a pattern, to apply to more objects. For example, the numbers 3, 5, and 7 are all of the form $2k + 1$ for some natural number k. That is, $3 = 2(1) + 1$, $5 = 2(2) + 1$, and $7 = 2(3) + 1$. We can use *inductive* reasoning to extend this property to all odd numbers. By noticing a pattern, we can convince ourselves that all odd numbers have this property. But, this is not sufficient proof of that fact. Luckily, mathematicians have a strategy for proving statements like this.

Suppose you have a magic, infinitely tall ladder. It has no top step, but it has a first step labeled "Step 0." It is followed by "Step 1," "Step 2," and so on. I claim that I can prove to you that every step on this ladder is reachable, despite the fact that there are infinitely many. This is how I do it:

1. I can definitely get from the ground to Step 0.
2. If I'm on any one step of the ladder, call it Step N, I can definitely get to the next step, Step N+1.

How does this proof work? Well, suppose I wanted to get to Step 1,000. Well according to what I've said, I can definitely get there from Step 999. And I can get to Step 999 from Step 998, which I can get to from Step 997, and so on. Eventually I find that to reach Step 1,000, all I really need to do is get to Step 0, and then I can go wherever I want! And since I can definitely get to Step 0, I can get to any step that I want!

This is what we call *mathematical induction.* It's the tool we use to prove that patterns we notice really do go on forever! Instead of a magic ladder with steps, we use the natural numbers! For example, earlier we noticed that it seems like all odd numbers can be written as $2k + 1$ for some other natural number k. How can we prove this using mathematical induction?

1. Prove that it works for the first odd number.
2. Prove that if it works for one odd number, it will definitely work for the next consecutive odd number.

Let's try it! On the next page, we'll walk through a proof by mathematical induction.

(a) Prove that the property holds for the first odd number. That is, show that the number 1 can be written as $2k + 1$ for some natural number k.

(b) Suppose the property holds for some odd number N. What does this mean about N? What is another form in which we could write N?

(c) We want to prove that IF the property holds for this odd number N, then the property MUST hold for the next odd number. How can we write the next odd number after N as a mathematical expression?

(d) What property must the next odd number after N satisfy?

(e) Prove that this next odd number after N does indeed satisfy our property. Remember that we have ASSUMED that N satisfies the property, regardless of whether it actually does or not for a particular value of N.